A DIRECT DEFINITION OF LOGARITHMIC DERIVATIVE.*

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1. The Usual Definitions. The logarithmic derivative of a function $y = f(x)$ is usually defined either as the derivative of the Napierian logarithm of y:

$$\frac{d \log_e y}{dx} = \frac{d \log_e f(x)}{dx}; \tag{1}$$

or by means of the resulting formula:

$$\frac{dy}{dx} \div y = f'(x) \div f(x). \tag{2}$$

These definitions suffer from the fact that the logarithm is defined only when $f(x)$ is positive, though (2) may be used independently of (1) when $f(x)$ is negative.

2. A Direct Definition. The importance of logarithmic differentiation in elementary work arises from its meaning as the *relative rate of increase* of y with respect to x. From this standpoint, the intervention of logarithms in the definition is wholly accidental and extraneous.

In order to define directly the relative rate of increase, we may divide the actual increase Δy by the increase in x, Δx, divide this ratio $\Delta y/\Delta x$ by some average value of $f(x)$ in the range Δx, and take the limit of this quotient as Δx approaches zero. Denoting the relative rate of increase by r_r, we may write

$$r_r = \lim_{\Delta x = 0} \left\{ \frac{\Delta y}{\Delta x} \div [\text{Average Value of } f(x)] \right\}. \tag{3}$$

The average value of $f(x)$ to be used may be selected in a variety of ways; that which seems most appropriate is the usual expression:

$$[\text{Average Value of } f(x)] = \frac{1}{\Delta x} \int_{x=a}^{x=a+\Delta x} f(x) dx,$$

where $x = a$ is the value of x at which the relative rate is to be found. Then (3) becomes

$$r_r = \lim_{\Delta x=0} \left\{ \frac{\Delta y}{\Delta x} \div \left[\frac{1}{\Delta x} \int_{x=a}^{x=a+\Delta x} f(x) dx \right] \right\},$$

whence, simplifying and replacing Δy by $f(a + \Delta x) - f(a)$, we obtain

$$r_r = \lim_{\Delta x=0} \left[\frac{f(a + \Delta x) - f(a)}{\int_a^{a+\Delta x} f(x) dx} \right]. \tag{4}$$

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* Read before the Southwestern Section of the American Mathematical Society, November, 1912.

Let $\phi(x)$ be any indefinite integral of $f(x)$; then

$$\int f(x)dx = \phi(x), \qquad \int_a^{a+\Delta x} f(x)dx = \phi(a + \Delta x) - \phi(a),$$

and (4) becomes

$$r_r = \lim_{\Delta x=0} \left[\frac{f(a + \Delta x) - f(a)}{\phi(a + \Delta x) - \phi(a)} \right], \tag{5}$$

which we shall adopt as *the definition of the relative rate of increase of $f(x)$, or the logarithmic derivative of $f(x)$, with respect to x.*

3. Identification with the Usual Definition. At least when $f(x)$ is positive, the definition (5) coincides with the usual definition (1); and (5) coincides with the definition (2) whenever (2) has a meaning.

For, since

$$\frac{f(a + \Delta x) - f(a)}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\phi(a + \Delta x) - \phi(a)} \cdot \frac{\phi(a + \Delta x) - \phi(a)}{\Delta x} \tag{6}$$

and since the limit of $[\phi(a + \Delta x) - \phi(a)]/\Delta x$ surely exists and is equal to $f(a)$, if $f(x)$ is continuous at $x = a$, we have

$$f'(a) = r_r \cdot f(a) \tag{7}$$

provided $f'(a)$ exists.

Conversely, if r_r, as defined by (5), exists, and if $f(x)$ is continuous at $x = a$, it follows from (6) that $f'(a)$ exists.

4. The Law of the Mean. If $f(x)$ is continuous, and if r_r exists, the derivative $f'(x)$ exists; moreover $\phi'(x)$ exists and is equal to $f(x)$; hence, by the generalized law of the mean,*

$$\frac{f(a + \Delta x) - f(a)}{\phi(a + \Delta x) - \phi(a)} = \frac{f'(c)}{\phi'(c)} = \frac{f'(c)}{f(c)}, \tag{8}$$

where c lies between a and $a + \Delta x$.

The expression on the left, $[f(a + \Delta x) - f(a)] \div [\phi(a + \Delta x) - \phi(a)]$, may be called the average relative rate over the interval Δx; with this notation, the equation (8) may be stated as follows: *The average relative rate over the interval Δx is precisely equal to the instantaneous relative rate at some point in that interval.*

This statement, which is the analogon to the law of the mean for ordinary derivatives, may be used to prove directly several important theorems.

Thus, *if the logarithmic derivative is zero identically, it follows from* (8) *that $f(x)$ is a constant.*

Again, suppose that the logarithmic derivatives of two functions $f(x)$ and $F(x)$ are identically equal; then

$$0 = \frac{f'(x)}{f(x)} - \frac{F'(x)}{F(x)} = \frac{f'(x)F(x) - F'(x)f(x)}{[F(x)]^2} \div \frac{f(x)}{F(x)}.$$

* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 8; it is vital that c is the same in numerator and denominator.

The extreme right-hand member is, however, the logarithmic derivative of $f(x)/F(x)$; hence by the preceding theorem, $f(x)/F(x)$ is constant, or $f(x) = k \cdot F(x)$: *If two functions have the same logarithmic derivative, their quotient is a constant*; or *If the logarithmic derivative of a function is given, that function is determined except for a constant factor.*

These are the fundamental theorems concerning logarithmic derivatives; they are precisely analogous to the corresponding theorems for ordinary derivatives.

5. Discontinuity. It has been shown above that $f'(x)$ exists whenever the relative rate r_r defined by (5) exists, provided only that $f(x)$ is continuous. It is, however, possible that $f'(x)$ and therefore also r_r is discontinuous.

It might appear that (8) precludes such a possibility, for the left side approaches r_r, and c approaches a as Δx approaches zero. But c does not necessarily take on all values; hence the approach of c to a may be only through a set of special values of x.

That the possibility just mentioned actually does occur is manifested by the example

$$f(x) = 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x} + 1, \qquad x \neq 0; \qquad f(0) = 1;$$

which has an indefinite integral:

$$\phi(x) = x^4 \sin\frac{1}{x} + x, \qquad x \neq 0; \qquad \phi(0) = 0.$$

Its derivative is

$$f'(x) = 12x^2 \sin\frac{1}{x} - 5x \cos\frac{1}{x} + \sin\frac{1}{x}, \qquad x \neq 0; \qquad f'(0) = 0;$$

which is discontinuous. Its logarithmic derivative may be derived either by (2) or by (5), and its value when $x = 0$ is zero.

6. Conclusion. The facts and illustrations given above parallel completely the fundamental theorems and illustrations usually given for ordinary derivatives. In deducing them, although the name *logarithmic* derivative has been used, the notion of logarithms has not been employed, and it has been shown that the ideas themselves are independent of the concept of logarithms.

TWO GEOMETRICAL APPLICATIONS OF THE METHOD OF LEAST SQUARES.

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The two problems discussed in the present paper are taken from a recent work by Vahlen entitled "Konstruktionen und Approximationen."* It seems to me that they are sufficiently interesting and important to merit special attention.

* Leipzig, 1911, pages 125, 126.